Pythagoras
2021-1-RO01-KA220-HED-000032258
Funded by the European Union

Mini-PBL example

Teaching Guide for Teachers

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Develop thinking strategies to solve real life problems
Explore, analyse, and apply mathematical ideas
Estimate reasonably and demonstrate fluent, flexible, and strategic thinking about graphs
Model with mathematics in situational contexts
Think creatively and with curiosity and wonder when exploring problems

Understanding and solving

Develop, demonstrate, and apply conceptual understanding of mathematical ideas through story, inquiry, and problem solving
Visualize to explore and illustrate mathematical concepts and relationships
Apply flexible and strategic approaches to solve problems
Solve problems with persistence and a positive disposition
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	• Engage in problem-solving experiences connected with real-life examples. **Communicating and representing** • Explain and justify mathematical ideas and decisions in many ways • Represent mathematical ideas in concrete, pictorial, and symbolic forms • Use mathematical vocabulary and language to contribute to discussions in the classroom • Take risks when offering ideas in classroom discourse **Connecting and reflecting** • Reflect on mathematical thinking • Connect mathematical concepts with each other, other areas, and personal interests • Use mistakes as opportunities to advance learning • Incorporate First Peoples worldviews, perspectives, knowledge, and practices to make connections with mathematical concepts
ICT tools to be used	Available Computer Algebra Systems: Mathematica
Context: project statement	Travellers in cars are exposed to fatigue and unwanted vibrations that affect their health. Driver fatigue affects his performance and reactions, which increases the risk of a traffic accident. Undesirable vibrations, the frequency of which depends on the speed of the car, occur when the car drives on an uneven road (a road with a different profile) in the vertical direction (causing floating), in the longitudinal direction (causing rocking) and in the transverse direction (causing swinging). They act on a person through the floor, seat and steering wheel of the car. In cars, we consider oscillation in the vertical direction (vertical acceleration), which affects the driver and passengers, to be a decisive factor. For this direction of oscillation, it is necessary to avoid the frequency band in the range of 4-8 Hz (the natural frequency of the human organism in the abdomen), which causes nausea. The safety of the car and the comfort of passengers are improved by the suspension system of the car and properly designed damping. Their task is to capture and dampen vertical forces when the car drives over an uneven road. From the practical point of view - the safe driving of passengers on an uneven road - it is important that the movement of the driver and passengers in the car seats when driving on an uneven road stabilizes as soon as possible, and adapts to the shape of the road so that the dangerous acceleration of the car's suspended masses is dampened as soon as possible. Several models of the car - quarter, half and full are used in order to simulate the oscillation of the car, it's non spring-loaded and spring-loaded masses when driving on an uneven road at different speeds. Each model works with an excitation signal from the road (kinematic excitation). Non spring-loaded masses of a car are wheel, tire, axle, brakes, ... Spring-loaded masses of a car are body, driver, passengers, seats, cargo, ...

The behavior of the car when driving on an uneven road allows us to know the solution of the equations of motion for the corresponding model of the car in the physical and state space.

Tasks and problems

TASK

A sports car weighing *m* = 1000 kg moves along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function *u*(*t*) = sin(*t*). Equivalent values of the stiffness and damping of the car suspension are *k* = 2000 N/m and *b* = 2000 Ns/m. Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (of the forced oscillation system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x}+b\left(\dot{x}-\dot{u}\right)+k\left(x-u\right)=0$$
$$x(0)=0$$
$$\dot{x}(0)=30$$

where $x = x\,(t)$ – a deflection of a car moving on an uneven road

$x' = x'(t)\ ,\ x'' = x''(t)\ ,\ u = u(t)$

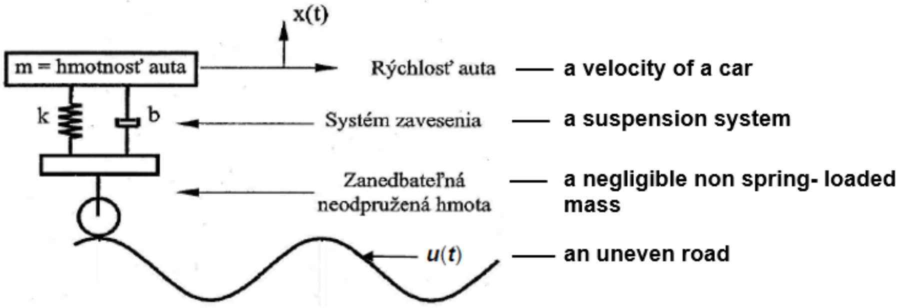


Fig. Quarter mechanical model (with 1 degree of freedom) of a car moving on uneven road

Task 1.

Transform the motion equation into the state space.
Solve next tasks in this space.

***Solution guide*:**

Transform the motion equation into a system of linear differential equations of the first order in the form $\bar{\boldsymbol{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\bar{\boldsymbol{z}}(\boldsymbol{t}) + \bar{\boldsymbol{f}}(\boldsymbol{t})$ along with Cauchy initial conditions.
First adjust the motion equation in the form $x'' = f(t, x, x')$.
Then realize its transformation into the state space by reducing the order of the differential equation using $x(t) = z_1\,(t)\ ,\ x'(t) = z_2\,(t)$.

	Solution:

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{k}{m}u + \frac{b}{m}\dot{u}$$

Transformation into the state space

$$\dot{z}_1 = z_2 \qquad ; \quad z_1(0) = 0$$

$$\dot{z}_2 = -\frac{k}{m}z_1 - \frac{b}{m}z_2 + \left(\frac{k}{m}u + \frac{b}{m}\dot{u}\right) \quad ; \quad z_2(0) = 30$$

and in the matrix form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k}{m}u + \frac{b}{m}\dot{u} \end{pmatrix} \; ; \; \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$$

$$\overline{z}'(t) = A.\overline{z}(t) + \overline{f}(t) \; ; \; \overline{z}(0) = (0,30)^T$$

Insert input values and solve the system of two linear differential equations with constant coefficients

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\sin(t) + 2\cos(t) \end{pmatrix} \; ; \; \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$$

Task 2.
What is the difference between solutions of the equation of motion in the physical space and in the state space?

Solution:
Solution of the equation of motion

- in the physical space
  $x\ (t)$ – only deflection of a moving car
- in the state space
  $z_1(t) = \ x(t)$ – a deflection of a moving car
  $z_2(t) = x'\ (t)$ – a velocity of a moving car

Task 3.
Investigate the stability of the system with forced oscillation.
(Use the eigenvalues of matrix A of the system $\overline{\boldsymbol{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\overline{\boldsymbol{z}}(\boldsymbol{t}) + \overline{\boldsymbol{f}}(\boldsymbol{t})$.)

Solution guide:

The forced oscillation system is stable (i. e. a forced oscillation of the non spring-loaded masses of a car are dampened) when all eigenvalues of matrix A are $\lambda \in C \wedge a < 0\,, \lambda = a + ib$.

After inserting solution in the form $\bar{z}(t) = \bar{v}e^{\lambda t}$ into the system $\bar{z}'(t) = A\,\bar{z}(t)$, the solution of the resulting system $A\,\bar{v} = \lambda\,\bar{v}$ will be an eigenvector $\bar{v}$ (an eigen shape of an oscillation) belonging to λ.
Algebraically manipulate this system in order to obtain a homogeneous system of equations.
Calculate all eigenvalues λ ($\lambda = \Omega^2$, where Ω is an eigen frequency of an oscillation) of the matrix of the homogeneous system from the requirement for its non-trivial solution ($\bar{v} \neq \bar{0}$).

Solution:

$$A \cdot \bar{v} = \lambda \cdot \bar{v} \Leftrightarrow (A - \lambda E) \cdot \bar{v} = \bar{0} \rightarrow \det(A - \lambda E) = 0$$

$$(A - \lambda E) = \begin{pmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda E) = 0 \Leftrightarrow 2 + 2\lambda + \lambda^2 = 0$$

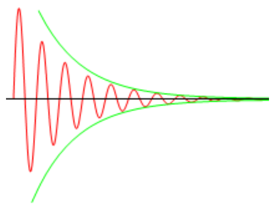
$$\lambda \to -1 - i$$
$$\lambda \to -1 + i$$

Conclusion.

The forced oscillation system is stable because all $\lambda \in C \wedge a < 0$.

Comment.
Look at the solution

$\bar{z}(t) = \bar{v}e^{\lambda t} = \bar{v}e^{(a+ib)t} = \bar{v}e^{at}\big(cos(bt) + isin(bt)\big)$.



Task 4.

Compare the solution of the characteristic equation of the homogeneous motion equation with eigenvalues of matrix *A.* Explain.

Solution:
The characteristic equation of the homogeneous motion equation

$$x'' + 2x' + 2x = 0$$

is

$$r^2 + 2r + 2 = 0$$

Solutions of the characteristic equation $r_1 = -1 + i$, $r_1 = -1 + i$

Eigenvalues of matrix *A* $\lambda_1 = -1 + i$, $\lambda_2 = -1 + i$

Explanation – individual text inserted by students.

Task 5.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec.

Solution guide:

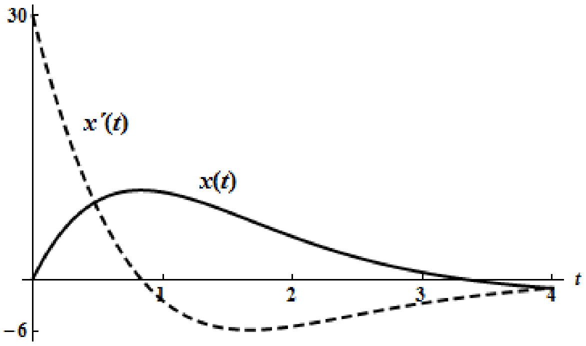
We are looking for an analytical solution of the system

$\bar{z}'(t) = A\,\bar{z}(t) + \bar{f}(t)\,,\ \bar{z}(0) = (0{,}30)^T$ for time $t \in [0, 4]$ in sec.

Solution:

$$x(t) = -\frac{2\,\mathrm{Cos}[t]}{5} + \frac{2}{5}\,e^{-t}\,\mathrm{Cos}[t] + \frac{6\,\mathrm{Sin}[t]}{5} + \frac{146}{5}\,e^{-t}\,\mathrm{Sin}[t]$$

$$\dot{x}(t) = \frac{6\,\mathrm{Cos}[t]}{5} + \frac{144}{5}\,e^{-t}\,\mathrm{Cos}[t] + \frac{2\,\mathrm{Sin}[t]}{5} - \frac{148}{5}\,e^{-t}\,\mathrm{Sin}[t]$$



Task 6.
To solve the system $\bar{\mathbf{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\bar{\mathbf{z}}(\boldsymbol{t}) + \bar{\boldsymbol{f}}(\boldsymbol{t}),\ \bar{\mathbf{z}}(\mathbf{0}) = (\mathbf{0}, \mathbf{30})^{\boldsymbol{T}}$ we will use numerical methods.

Analyse conditions of the numerical solvability of this system (the existence and the uniqueness of the solution) under which it is solvable on interval [0; 4].

Solution:

Individual text inserted by students.

NOTE: Solve tasks 7.– 11. using software Mathematica.

Task 7.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by Euler´s method with step $h = 0.1$.

Solution guide:
We are looking for an approximate solution of the system

$\bar{z}'(t) = A\,\bar{z}(t) + \bar{f}(t)\,,\ \bar{z}(0) = (0{,}30)^T$ for time $t \in [0, 4]$ in sec, applying numerical method.

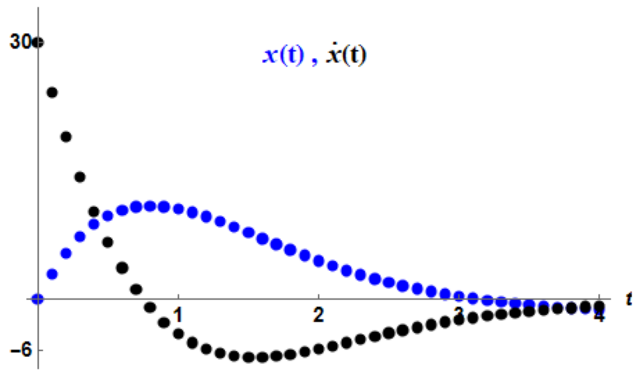
Solution:
The iteration scheme for Euler´s method

$$\bar{k} = A \cdot \bar{z}_i + \bar{f}(t_i)$$
$$\bar{z}_{i+1} = \bar{z}_i + h \cdot \bar{k}$$
$$t_{i+1} = t_i + h$$
$$i = 0, 1, \ldots, n$$

Numerical solutions (We only list every 5th solution)

i	t	$z_1 = x_E$	$z_2 = \dot{x}_E$
0	0	0	30
5	0.5	9.7768418	6.7154412
10	1.	10.566112	-4.0296232
15	1.5	7.7809817	-6.7590516
20	2.	4.5123272	-5.8094278
25	2.5	1.9778044	-3.9461663
30	3.	0.33728205	-2.3942647
35	3.5	-0.63961185	-1.3911753
40	4.	-1.1977435	-0.74981349

Graphical solutions



Task 8.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by the Runge-Kutta´s method of the 4th order with step $h = 0.1$.

Solution guide:
We are looking for an approximate solution of the system
$\bar{z}'(t) = A\,\bar{z}(t) + \bar{f}(t)$, $\bar{z}(0) = (0, 30)^T$ for time $t \in [0, 4]$ in sec,
applying numerical method.

Solution:
The iteration scheme for Runge-Kutta´s method of the 4th order

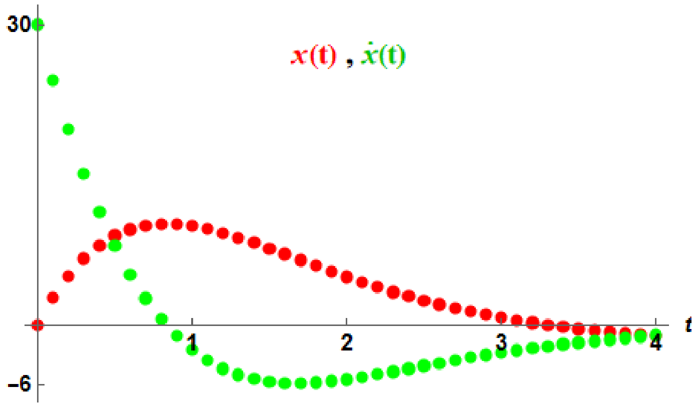
$$\bar{k}_1 = A\cdot\bar{z}_i + \bar{f}(t_i)$$
$$\bar{k}_2 = A\cdot\left(\bar{z}_i + \frac{h}{2}\cdot\bar{k}_1\right) + \bar{f}\left(t_i + \frac{h}{2}\right)$$
$$\bar{k}_3 = A\cdot\left(\bar{z}_i + \frac{h}{2}\cdot\bar{k}_2\right) + \bar{f}\left(t_i + \frac{h}{2}\right)$$
$$\bar{k}_4 = A\cdot\left(\bar{z}_i + h\cdot\bar{k}_3\right) + \bar{f}(t_i + h)$$
$$\bar{k} = \frac{\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4}{6}$$

$$\bar{z}_{i+1} = \bar{z}_i + h\cdot\bar{k}$$
$$t_{i+1} = t_i + h$$
$$i = 0,1,\ldots,n$$

Numerical solutions (We only list every 5th solution)

i	t	$z_1 = x_{RK4}$	$z_2 = \dot{x}_{RK4}$
0	0	0	30
5	0.5	8.9281666	7.9672212
10	1.	9.9122941	-2.4536062
15	1.5	7.674066	-5.649672
20	2.	4.8284078	-5.4002166
25	2.5	2.4467586	-4.0700151
30	3.	0.75076544	-2.759002
35	3.5	-0.36698534	-1.7649179
40	4.	-1.0562481	-1.0215761

Graphical solutions



Task 9.

Compare accuracy of used numerical methods by calculation and graphically. (Use global trunction error of a numerical method.)

Solution guide:

Global trunction error of a numerical method is $|x_i(t) - x_i|$,

where $x_i(t)$ – an analytical solution, x_i – an approximate solution

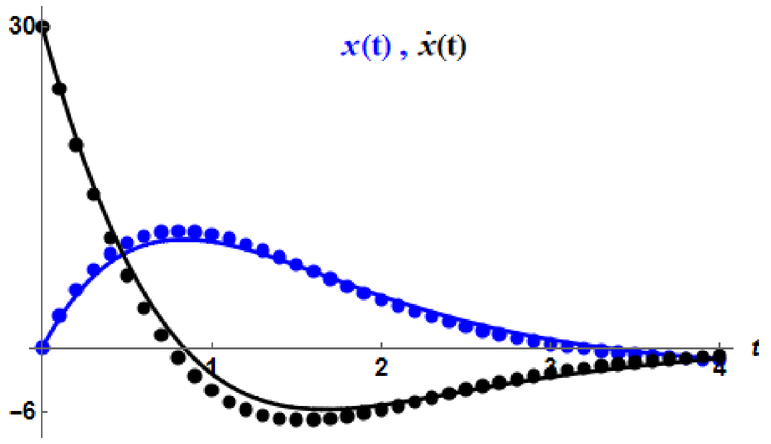
Solution:

Comparison by calculation (We only list every 5th solution)

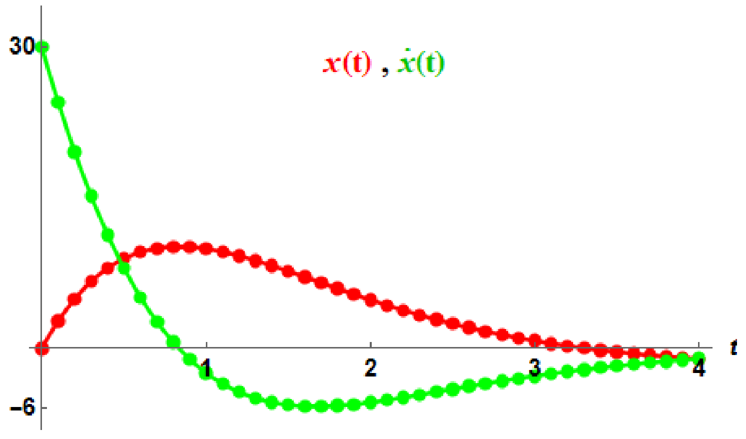
i	t	$\lvert x(t)-x_E\rvert$	$\lvert \dot{x}(t)-\dot{x}_E\rvert$	$\lvert x(t)-x_{RK4}\rvert$	$\lvert \dot{x}(t)-\dot{x}_{RK4}\rvert$
0	0	0.	0.	0.	0.
5	0.5	0.84869224	1.2518389	0.000017103898	0.000059030232
10	1.	0.65381293	1.576066	4.9829304×10^{-6}	0.000049065915
15	1.5	0.10688962	1.109394	0.000026081243	0.000014363026
20	2.	0.31611341	0.40919656	0.00003286521	0.000014643786
25	2.5	0.46898199	0.12387642	0.000027730321	0.000027660012
30	3.	0.41350121	0.36476462	0.000017824845	0.000027296591
35	3.5	0.27263535	0.37376304	8.8493427×10^{-6}	0.000020429769
40	4.	0.14149862	0.27177548	3.2458854×10^{-6}	0.000012823749

Graphical comparison

Euler´s method versus the analytical solution



Runge-Kutta´s method of the 4th order versus the analytical solution



Task 10.

Calculate and compare values of the accelerations at the time t = 3 sec. Use Central difference formula of the order $O(h^2)$.

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Solution:

- The exact value of the acceleration

$$\ddot{x}(t) = \frac{2\,\mathrm{Cos}[t]}{5} - \frac{292}{5} e^{-t}\,\mathrm{Cos}[t] - \frac{6\,\mathrm{Sin}[t]}{5} + \frac{4}{5} e^{-t}\,\mathrm{Sin}[t]$$

$$\ddot{x}(t=3) = 2.318747\ldots \doteq 2.31875$$

- Approximate values of the accelerations

Central difference formula of the order $O(h^2)$

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

Euler´s method

i	t	x_E
28	2.8	0.89795115
29	2.9	0.6031926
30	3.	0.33728205
31	3.1	0.097855585
32	3.2	-0.11740868

Runge-Kutta´s method

i	t	x_{RK4}
28	2.8	1.3507571
29	2.9	1.0384891
30	3.	0.75076544
31	3.1	0.48622766
32	3.2	0.24349775

Comparison

	Euler´s m. $\ddot{x}_E$	Runge-Kutta´s m. $\ddot{x}_{RK4}$	Analytical solution $\ddot{x}(t)$
values	2,64841	2,31859	2,31875
$\lvert \ddot{x}(t) - \ddot{x} \rvert$	0,32966	0,00016	–
$\lvert \ddot{x}_E - \ddot{x}_{RK4} \rvert$	0,3295		–
$\ddot{x}_E = K\,\ddot{x}_{RK4}$	$K \approx 2061$		–
$\ddot{x}_{RK4} = \tilde{K}\,\ddot{x}_E$	$\tilde{K} \approx 0{,}0005$		–

Task 11.

At what times $t \in [0, 4]$ in sec it is possible to use the Non-symmetric backward formula of the order $O(h^2)$ to calculate the approximate values of accelerations? Give your reasons.

Solution guide:

Non-symmetric formula backward of the order $O(h^2)$

$$f''(x) \approx \frac{2f(x) - 5f(x-h) + 4f(x-2h) - f(x-3h)}{h^2}$$

Solution:

$t \in [0{,}3\,;\ 4]$

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Outcomes expected

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- Graphics fitting the solution;
- Numerical results explained and put in context;
- Capture of ICT tools solutions used;

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	- Sequence of steps followed; - Remark computations done by hand and done by ICT tools; - Provide complete answer to questions; - All the results must be presented in the context of the problem;
Guide for Learning	At the beginning of the course, the students need guides on new activities, and feel your support on a well-structured pack of suggestions on how to address the problems posted. Namely: - Read carefully the problem statement and the tasks posted. Always maintain a global view of all the projects. - Identify, or try to do a first draft match, the content units of your lecture notes involved in every task. - Take your lecture notes open and review before starting to solve the problems. - Match output expected with the tasks posted, at least as first draft approach. - Follow the order of the tasks, try to increase the knowledge of the problem while you are solving the activities. - Always think that maybe there are different ways to solve a problem. - Use ICT tools to avoid hard computations and check your solutions are correct in different ways if possible. - The solutions are always part of a context, expressing such a final solution totally integrated in the problem posted. - Be sure you answer the complete questions. - Always try to solve the questions by yourself. - If the project can be done in groups, discuss with the groups the proposed problem, to confirm and detect fails or weaknesses, confront strategies, discuss presentation format, etc. Working in groups doesn't mean work less but work better.
Guide for Teaching	Some hints needed to present and launch the mini-PBL to students - Do a small Introduction concerning Energy consumption, added to the Climate Change crisis we are currently living in. - Do a small introduction about the relations between power and energy, with the basic equations. - Students will form groups of 4 students and solve the mini-PBL using the eduScrum methodology. - The students should do each exercise in a sequential order, starting from Task 1. - The students should be able to thoroughly read and interpret the numerical results from a mathematical and the real-life example point of view. They should include also a discussion of the climate change crisis and enumerate some strategies they could apply at home or even at university to save resources, namely reduce energy consumption. They should also mention how this mini-PBL helps them identify the Sustainable Development Goals 4, 7 and 9.
Assessment	- Final report; - Oral presentation; - Peer-assessment: students will apply peer-assessment for their periodic performance using online peer assessment tools used

	and available at the respective institution.
Others: References	Starek, L. Kmitanie s riadením. Vydavateľstvo STU, Bratislava, 2009. Starek, L. Kmitanie mechanických sústav. Vydavateľstvo STU, Bratislava, 2006.

Learning Guide for Students

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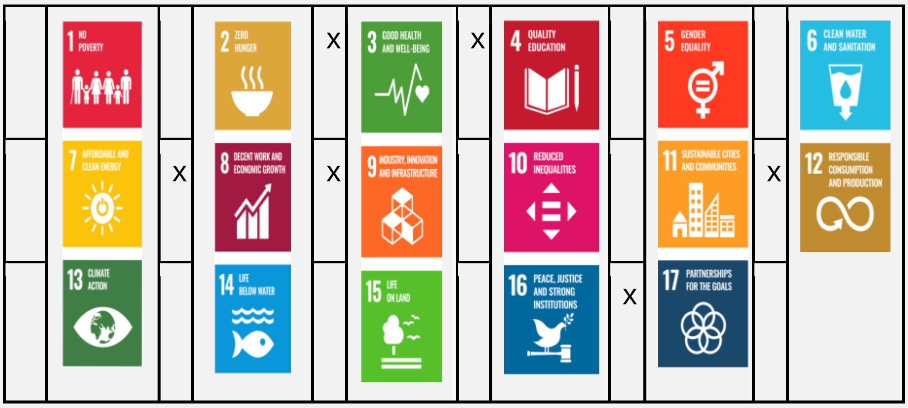
SDG attended	Using this UN graphics, we mark such SDG which this project works. 
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Sessions	4 sessions of 100 min
Hours of autonomous work	100 min
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	- Represent mathematical ideas in concrete, pictorial, and symbolic forms - Use mathematical vocabulary and language to contribute to discussions in the classroom - Take risks when offering ideas in classroom discourse **Connecting and reflecting** - Reflect on mathematical thinking - Connect mathematical concepts with each other, other areas, and personal interests - Use mistakes as opportunities to advance learning - Incorporate First Peoples worldviews, perspectives, knowledge, and practices to make connections with mathematical concepts
ICT tools to be used	Available Computer Algebra Systems: Mathematica
Context: project statement	Travellers in cars are exposed to fatigue and unwanted vibrations that affect their health. Driver fatigue affects his performance and reactions, which increases the risk of a traffic accident. Undesirable vibrations, the frequency of which depends on the speed of the car, occur when the car drives on an uneven road (a road with a different profile) in the vertical direction (causing floating), in the longitudinal direction (causing rocking) and in the transverse direction (causing swinging). They act on a person through the floor, seat and steering wheel of the car. In cars, we consider oscillation in the vertical direction (vertical acceleration), which affects the driver and passengers, to be a decisive factor. For this direction of oscillation, it is necessary to avoid the frequency band in the range of 4-8 Hz (the natural frequency of the human organism in the abdomen), which causes nausea. The safety of the car and the comfort of passengers are improved by the suspension system of the car and properly designed damping. Their task is to capture and dampen vertical forces when the car drives over an uneven road. From the practical point of view - the safe driving of passengers on an uneven road - it is important that the movement of the driver and passengers in the car seats when driving on an uneven road stabilizes as soon as possible, and adapts to the shape of the road so that the dangerous acceleration of the car's suspended masses is dampened as soon as possible. Several models of the car - quarter, half and full are used in order to simulate the oscillation of the car, it’s non spring-loaded and spring-loaded masses when driving on an uneven road at different speeds. Each model works with an excitation signal from the road (kinematic excitation). Non spring-loaded masses of a car are wheel, tire, axle, brakes, ... Spring-loaded masses of a car are body, driver, passengers, seats, cargo, ... The behavior of the car when driving on an uneven road allows us to know the solution of the equations of motion for the corresponding model of the car in the physical and state space.

Tasks and problems

TASK

A sports car weighing *m* = 1000 kg moves along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function *u*(*t*) = sin(*t*). Equivalent values of the stiffness and damping of the car suspension are *k* = 2000 N/m and *b* = 2000 Ns/m. Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (of the forced oscillation system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$$

$$x(0)=0$$

$$\dot{x}(0)=30$$

where $x = x\,(t)$ – a deflection of a car moving on an uneven road

$x' = x'(t)\ ,\ x'' = x''(t)\ ,\ u = u(t)$

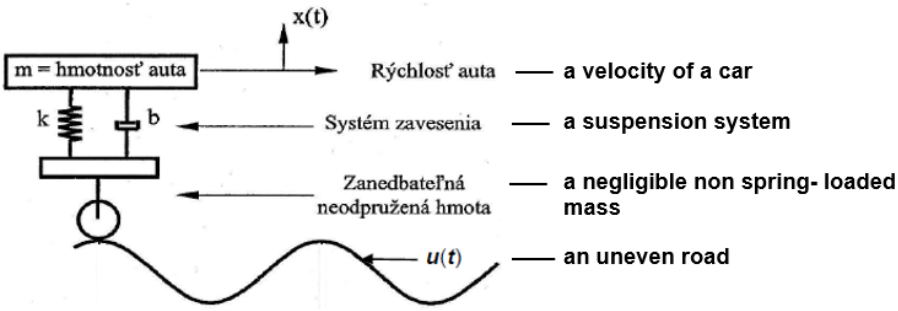

Fig. Quarter mechanical model (with 1 degree of freedom) of a car moving on uneven road

Task 1.

Transform the motion equation into the state space.
Solve next tasks in this space.

***Solution guide*:**

Transform the motion equation into a system of linear differential equations of the first order in the form $\bar{\boldsymbol{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\bar{\boldsymbol{z}}(\boldsymbol{t}) + \bar{\boldsymbol{f}}(\boldsymbol{t})$ along with Cauchy initial conditions.
First adjust the motion equation in the form $x'' = f(t, x, x')$.
Then realize its transformation into the state space by reducing the order of the differential equation using $x(t) = z_1\,(t)\,,\ x'(t) = z_2\,(t)$.

Task 2.

What is the difference between solutions of the equation of motion in the physical space and in the state space?

Task 3.

Investigate the stability of the system with forced oscillation.

	(Use the eigenvalues of matrix A of the system $\bar{\boldsymbol{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\bar{\boldsymbol{z}}(\boldsymbol{t}) + \bar{\boldsymbol{f}}(\boldsymbol{t})$.) ***Solution guide:*** The forced oscillation system is stable (i. e. a forced oscillation of the non spring-loaded masses of a car are dampened) when all eigenvalues of matrix *A* are $\lambda \in C \wedge a < 0\,, \lambda = a + ib$. After inserting solution in the form $\bar{z}(t) = \bar{v}e^{\lambda t}$ into the system $\bar{z}'(t) = A\,\bar{z}(t)$, the solution of the resulting system $A\,\bar{v} = \lambda\,\bar{v}$ will be an eigenvector $\bar{v}$ (an eigen shape of a oscillation) belonging to λ. Algebraically manipulate this system in order to obtain a homogeneous system of equations. Calculate all eigenvalues λ ($\lambda = \Omega^2$, where Ω is an eigen frequency of a oscillation) of the matrix of the homogeneous system from the requirement for its non-trivial solution ($\bar{v} \neq \bar{0}$). **Task 4.** Compare the solution of the characteristic equation of the homogeneous motion equation with eigenvalues of matrix *A.* Explain. **Task 5.** Find and display the deflection and velocity of a car moving on an uneven road in time *t* ∈ [0, 4] in sec. ***Solution guide:*** We are looking for an analytical solution of the system $\bar{z}'(t) = A\,\bar{z}(t) + \bar{f}(t)\,, \bar{z}(0) = (0{,}30)^T$ for time $t \in [0, 4]$ in sec. **Task 6.** To solve the system $\bar{\boldsymbol{z}}'(\boldsymbol{t}) = \boldsymbol{A}\,\bar{\boldsymbol{z}}(\boldsymbol{t}) + \bar{\boldsymbol{f}}(\boldsymbol{t}),\ \bar{\boldsymbol{z}}(\boldsymbol{0}) = (\boldsymbol{0}, \boldsymbol{30})^{\boldsymbol{T}}$ we will use numerical methods. Analyse conditions of the numerical solvability of this system (the existence and the uniqueness of the solution) under which it is solvable on interval [0; 4]. **NOTE:** Solve tasks 7.– 11. using software Mathematica. **Task 7.** Find and display the deflection and velocity of a car moving on an uneven road in time *t* ∈ [0, 4] in sec by the Euler´s method with step *h* = 0.1. ***Solution guide:*** We are looking for an approximate solution of the system $\bar{z}'(t) = A\,\bar{z}(t) + \bar{f}(t)\,,\ \bar{z}(0) = (0{,}30)^T$ for time $t \in [0, 4]$ in sec, applying numerical method.

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Task 8.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by the Runge-Kutta´s method of the 4th order with step $h = 0.1$.

Solution guide:
We are looking for an approximate solution of the system $\bar{z}\,'(t) = A\,\bar{z}(t) + \bar{f}(t)$, $\bar{z}(0) = (0,30)^T$ for time $t \in [0,4]$ in sec, applying numerical method.

Task 9.
Compare accuracy of used numerical methods by calculation and graphically. (Use global trunction error of a numerical method.)

Solution guide:
Global trunction error of a numerical method is $\left|x_i(t) - x_i\right|$,
where $x_i(t)$ – an analytical solution, x_i – an approximate solution

Task 10.
Calculate and compare values of the accelerations at the time $t = 3$ sec. Use Central difference formula of the order $O(h^2)$.

Task 11.
At what times $t \in [0,4]$ in sec it is possible to use the Non-symmetric backward formula of the order $O(h^2)$ to calculate the approximate values of accelerations? Give your reasons.

Solution guide:
Non-symmetric formula backward of the order $O(h^2)$:

$$f\,''(x) \approx \frac{2f(x) - 5f(x-h) + 4f(x-2h) - f(x-3h)}{h^2}$$

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Outcomes expected

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- Graphics fitting the solution;
- Numerical results explained and put in context;
- Capture of ICT tools solutions used;
- Sequence of steps followed;
- Remark computations done by hand and done by ICT tools;
- Provide complete answer to questions;
- All the results must be presented in the context of the problem;

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Guide for Learning

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- Read carefully the problem statement and the tasks posted. Always maintain a global view of all the projects.
- Identify, or try to do a first draft match, the content units of your lecture notes involved in every task.
- Take your lecture notes open and review before starting to solve the problems.
- Match output expected with the tasks posted, at least as first draft approach.
- Follow the order of the tasks, try to increase the knowledge of the problem while you are solving the activities.

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	- Always think that maybe there are different ways to solve a problem. - Use ICT tools to avoid hard computations and check your solutions are correct in different ways if possible. - The solutions are always part of a context, expressing such a final solution totally integrated in the problem posted. - Be sure you answer the complete questions. - Always try to solve the questions by yourself. - If the project can be done in groups, discuss with the groups the proposed problem, to confirm and detect fails or weaknesses, confront strategies, discuss presentation format, etc. Working in groups doesn't mean work less but work better.
Assessment	- Final report; - Oral presentation; - Peer-assessment: students will apply peer-assessment for their periodic performance using online peer assessment tools used and available at the respective institution.
Others: References	Starek, L. Kmitanie s riadením. Vydavateľstvo STU, Bratislava, 2009. Starek, L. Kmitanie mechanických sústav. Vydavateľstvo STU, Bratislava, 2006.

ANNEX 1: RUBRIC

Category	4=Excellent	3=Good	2=Low	1=Poor
Mathematical Concepts	Explanation shows complete understan- ding of the mathe- matical concepts used to solve the problem(s).	Explanation shows substantial understan- ding of the mathema- tical concepts used to solve the problem(s).	Explanation shows some understan- ding of the mathe- matical concepts needed to solve the problem(s).	Explanation shows very limited unders- tanding of the underlying concepts needed to solve the problem(s) OR is not written.
Mathematical Terminology and Notation	Correct terminology and notation are always used, making it easy to understand what was done.	Correct terminology and notation are usually used, making it fairly easy to understand what was done.	Correct terminolo- gy and notation are used, but it is sometimes not easy to understand what was done.	There is little use, or a lot of inappropriate use, of terminology and notation.
Strategy/Procedure	Typically, uses an efficient and effective strategy to solve the problem(s).	Typically, uses an effective strategy to solve the problem(s).	Sometimes uses an effective strategy to solve problems, but does not do it consistently.	Rarely uses an effective strategy to solve problems.
Completion	All problems are completed.	All but one of the problems are completed.	All but two of the problems are completed.	Several of the problems are not completed.
Mathematical Errors	90-100% of the steps and solutions have no mathematical errors.	Almost all (85-89%) of the steps and solu- tions have no mathematical errors.	Most (75-84%) of the steps and solu- tions have no mathematical errors.	More than 75% of the steps and solu- tions have mathe- matical errors.

Sources Checking				
Working with Others	Student was an engaged partner, listening to suggestions of others and working cooperatively throughout lesson.	Student was an engaged partner but had trouble listening to others and/or working cooperatively.	Student cooperated with others, but needed prompting to stay on-task.	Student did not work effectively with others.
Neatness and Organization	The work is presented in a neat, clear, organized fashion that is easy to read.	The work is presented in a neat and organized fashion that is usually easy to read.	The work is presented in an organized fashion but may be hard to read at times.	The work appears sloppy and unorganized. It is hard to know what information goes together.
Diagrams and Sketches	Diagrams and/or sketches are clear and greatly add to the reader's understanding of the procedure(s).	Diagrams and/or sketches are clear and easy to understand.	The work is presented in an organized fashion but may be hard to read at times.	Diagrams and/or sketches are difficult to understand or are not used.
CT tools used				