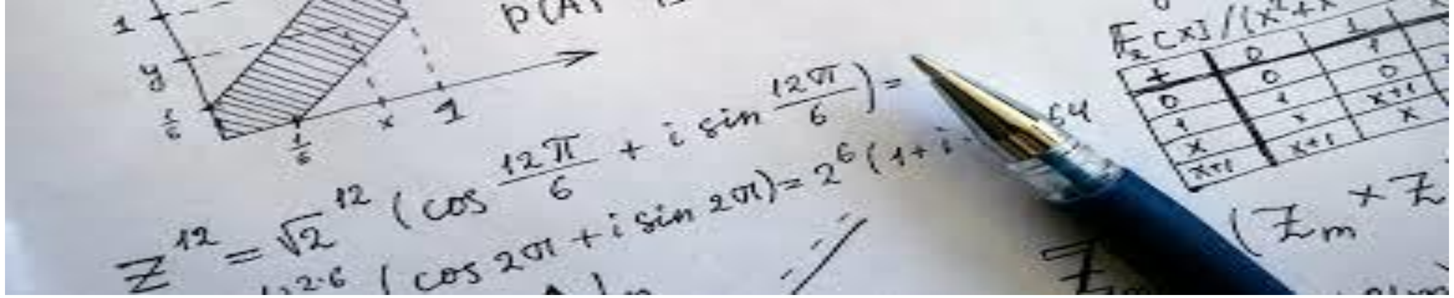

eduScrum at the Maths School

(reflection, evaluation and solution of problems from teamwork)

STU
SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

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Co-funded by the Erasmus+ Programme of the European Union

Innovative strategies for teaching and learning mathematics
11th - 15th March 2024 / Maths School in Tenerife

Pythagoras

eduScrum implementation at the Maths School

- ✓ eduScrum method was prepared by Slovak University of Technoloy in Bratislava,Slovakia for participants of the Maths School
- ✓ participants "'could try to solve" the same problems, that were solved by students in eduScrum experimente at FME STU in Bratislava, Slovakia
- ✓ participants worked in teams → 1team = 1 partner of the Pythagoras Project
- ✓ all teams solved the same problems
- ✓ the method eduScrum was realised in 2 topics for teamwork
  - **Calculus** – basic ***Mathematics I*** course
  - **Multiple integrals** – basic ***Mathematics II*** course

eduScrum implementation at the Maths School

✓ each topic for teamwork consisted of 5 problems

- $1^{st} – 4^{th}$ problem = standard mathematical problem
- 5^{th} problem = **applied problems** from the mechanical engineering field

✓ one of team was scrum master in the role of team leader

✓ during the teamwork participants cooperated together, though each one was responsible for solution of one from the problems distributed by team leader

✓ participants could achieve together max **37** points, whereas they could receive

- **Calculus** – max **12** points
- **Multiple integrals** – max **25** points

✓ participants had to solve 5 problems in each topic in about 60 minutes

SCORE SHEET

Team 1st – Math I / Calculus							
team leader	**participant (name)**	**Problem 1** (2 points)	**Problem 2** (2 points)	**Problem 3** (2 points)	**Problem 4** (2 points)	**Problem 5** (4 points)	**score**
	all participants					✓	
Max score: **12** points						$\sum score$	

Comment. Use a marker ✓ to mark the team leader and problem for a participant

Team 1st – Math II / Multiple Integrals							
team leader	**participant (name)**	**Problem 1** (3 points)	**Problem 2** (4 points)	**Problem 3** (5 points)	**Problem 4** (6 points)	**Problem 5** (7 points)	**score**
	all participants					✓	
Max score: **25** points						$\sum score$	

Comment. Use a marker ✓ to mark the team leader and problem for a participant

What problems were solved?

What are their solutions?

Math I / Calculus – 1st Problem

Problem 1.1 (2 points)

Let function f be given by formula $f: y = 3x.\ln^2 x$

a) Determine its domain of definition.

b) Find all zero points

c) Investigate parity of this function.

d) Find equations of all asymptotes to function graph.

***Solution*:**

$D(f) = (0, \infty)$

$x = 0 \notin D(f)$, $x = 1$

Function is not even nor odd.

$\lim_{x \to 0^+} f(x) = -3$, $\lim_{x \to 0^+} (-x) = 0$,

$\lim_{x \to \infty} (3.\ln^2 x) = \infty$

No asymptote to function graph exists.

Problem 1.2 (Theory)

Explain the usage of L´Hospital rule on example $\lim_{x \to 0^+} x^2 \ln x = ?$

Math I / Calculus – 2nd Problem

PROBLEM 2.1 (2 points)

Find domain of definition, intervals of monotonicity and local extrema of function

$$f : y = 3x.\ln^2 x$$

***Solution*:**

Domain of definition	$D(f) = (0, \infty)$	Increasing	$\left(0, \frac{1}{e^2}\right]$, $[1, \infty)$
$f'(x)$	$3\ln^2 x + 6\ln x$	Decreasing	$\left[\frac{1}{e^2}, 1\right]$
Stationary points	$x = 1$, $x = \frac{1}{e^2}$	Local max	$\left[\frac{1}{e^2}, \frac{12}{e^2}\right]$
		Local min	$[0, 1]$

Math I / Calculus – 2nd Problem

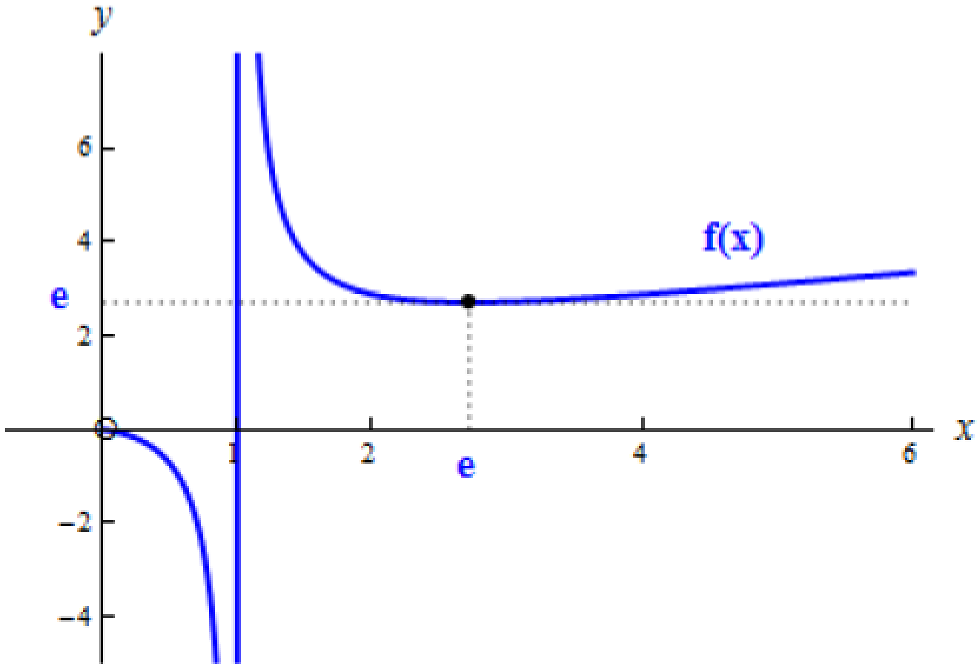
Problem 2.2 (Theory)

Graph of function $y = f(x)$, with the local minimum at the point $x = e$ of value $f(e) = e$ is presented in the figure.

a) Determine domain $D(f)$ and range $R(f)$

b) Write equation of asymptote without slope to graph of function and explain it.

c) Do exist some points of inflection of function f?
If yes, try to estimate their position at the function graph.



Math I / Calculus – 3rd Problem

PROBLEM 3.1 (2 points)

Determine domain of definition and identify intervals of convex and concave behaviour of function $f: y = 3x.\ln^2 x$. Find all points of inflection.

***Solution*:**

Domain of definition	$D(f) = (0, \infty)$	Points of inflection	$\left[\frac{1}{e}, \frac{3}{e}\right]$
$f''(x)$	$\frac{6}{x}.(\ln x + 1)$	Convex	$\left[\frac{1}{e}, \infty\right)$
		Concave	$\left[0, \frac{1}{e}\right)$

Problem 3.2 (Theory)

Let there exist a local extreme at the point $x = x_0$ from the domain of definition of function $f(x)$. What must be true for $f'(x_0)$? Explain on suitable examples of graphs sof various functions $f(x)$.

Math I / Calculus – 4th Problem

PROBLEM 4.1 (2 points)

Find maximum and minimum of function $f(x)=\frac{x^3+2}{(x+2)^3}$ **on interval** $\left[-\frac{1}{2},3\right]$.

Write equations of asymptotes without slope to the graph of function f.

***Solution*:**

Domain of definition	$R-\{-2\}$	Stationary points	$x=1$ $x=-1\notin\left(-\frac{1}{2},3\right)$
$f'(x)$	$\frac{6(x^2-1)}{(x+2)^4}$		
Global minimum	$\min\left\{f\left(-\frac{1}{2}\right);\ f(1);\ f(3)\right\}$	Global maximum	$\max\left\{f\left(-\frac{1}{2}\right);\ f(1);\ f(3)\right\}$
$x\in\left[-\frac{1}{2},3\right]$	$\min\left\{\frac{5}{9};\ \frac{1}{9};\ \frac{29}{125}\right\}=\frac{1}{9}=f(1)$	$x\in\left[-\frac{1}{2},3\right]$	$\max\left\{\frac{5}{9};\ =\frac{1}{9};\ \frac{29}{125}\right\}=\frac{5}{9}=f\left(-\frac{1}{2}\right)$

Asymptotes $\lim_{x\to-2^+} f(x)=-\infty$, $\lim_{x\to-2^-} f(x)=+\infty \rightarrow$ Equation of asymptote without slope: $x=-2$

Math I / Calculus – 4th Problem

Problem 4.2 (Theory)

Define point of inflection of function $f(x)$ and sketch its geometric interpretation.

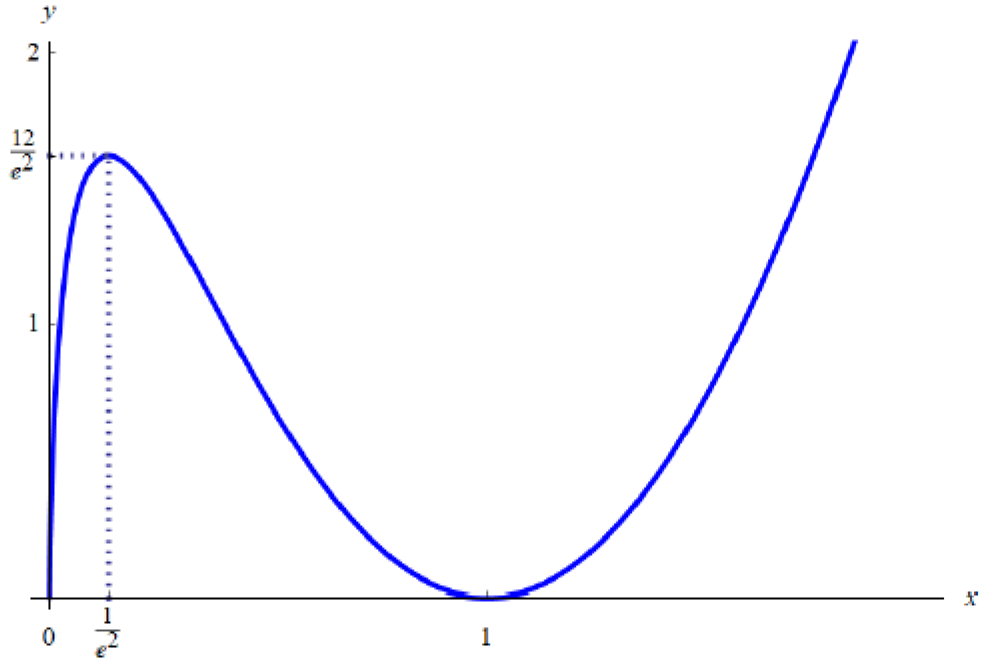
Math I / Calculus – 5th Problem

Problem 5.1 (2 points)

a) **Based on results from problems 1st , 2nd and 3rd sketch graph of function $f : y = 3x.\ln^2 x$ and determine its range $\mathcal{R}(f)$.**

***Solution*:**

$\mathcal{R}(f) = [0, \infty)$



Math I / Calculus – 5th Problem

Problem 5.1 (2 points)

b) Determine section x_m (Fig. 2) of beam located and loaded as ilustrated in Fig. 1, so that its bending moment $M(x)$ in section x_m be maximal, if

$$M(x)=\frac{ql}{12}x-\frac{q}{12l^2}x^4$$

where q – is relative load on the unit of length (a constant), l – is the length of beam (a constant).

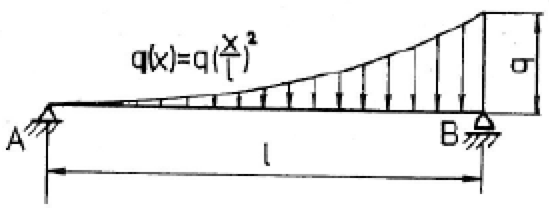


Fig.1

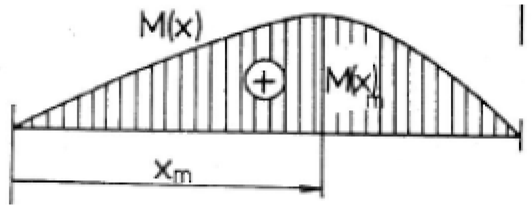


Fig.2

Solution:

$$M(x)=\frac{ql}{12}x-\frac{q}{12l^2}x^4$$

$$M'(x)=\frac{ql}{12}-\frac{q}{12l^2}4x^3=\frac{ql}{12}\left(1-\frac{4x^3}{l^3}\right)$$

$$M'(x)=0 \Leftrightarrow \left(1-\frac{4x^3}{l^3}\right)=0$$

$$l^3-4x^3=0 \Leftrightarrow x_m=x=\frac{l}{\sqrt[3]{4}}$$

$$M''(x)=-\frac{qx^2}{l^2},\ M''\left(\frac{l}{\sqrt[3]{4}}\right)=-\frac{q}{l^2}\left(\frac{l}{\sqrt[3]{4}}\right)^2=-\frac{q}{\left(2^{2/3}\right)^2}<0$$

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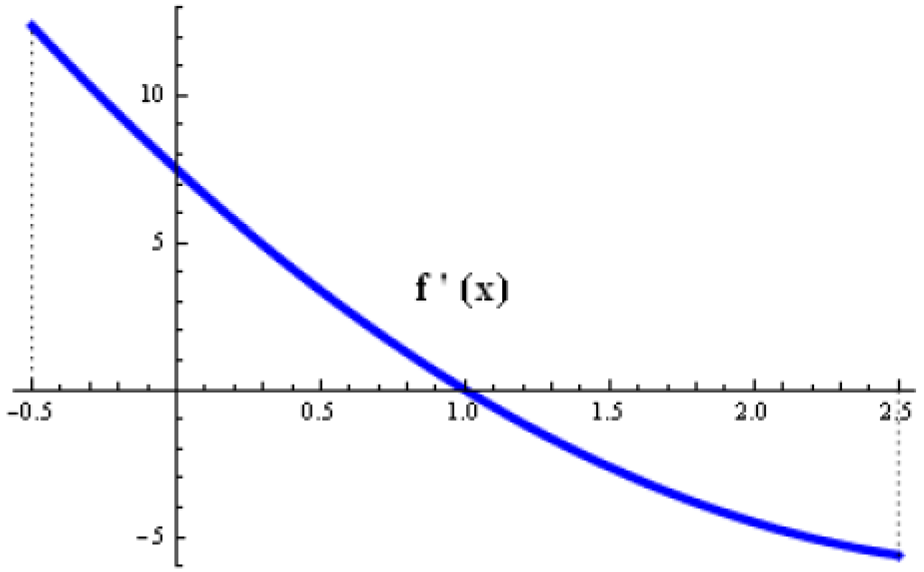
Math I / Calculus – 5th Problem

Problem 5.2 (Theory)

Graph of function $f\ '(x)$ on intervale (-0,5 ; 2,5) is given in the figure.

a) Can we claim that function $f(x)$ is strictly monote on interval (-0,5 ; 2,5) ?
Explain your answer.

b) Based on your solution sketch possible graph of function $f(x)$ on interval (-0,5 ; 2,5), if $f(-0,5)=f(2,5)=0$



Math II / Multiple Integrals – 1st Problem

Problem 1.1 (3 points)

Describe and sketch planar region bounded by graphs of functions $y = \sin x, y = \frac{2x}{\pi}$ and calculate its area by means of double integral.

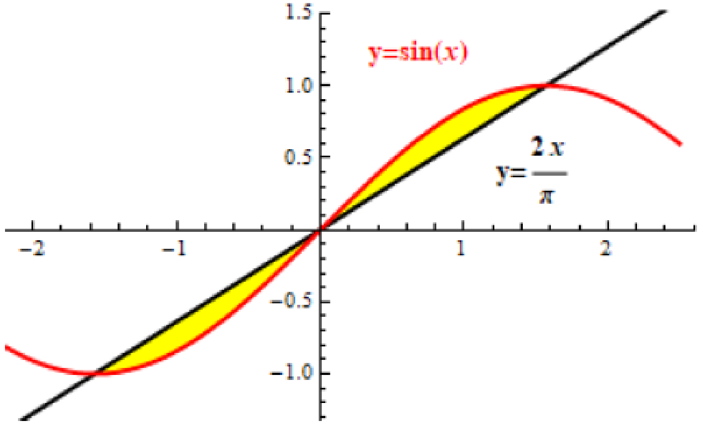
Solution:

$$M = M_1 \cup M_2$$

$$M_1 = \left\{[x,y] \in E^2; -\frac{\pi}{2} \le x \le 0, \sin x \le y \le \frac{2x}{\pi}\right\}$$

$$M_2 = \left\{[x,y] \in E^2; 0 \le x \le \frac{\pi}{2}, \frac{2x}{\pi} \le y \le \sin x\right\}$$

$$P_{M_1} = P_{M_2} \ , \ P_M = 2P_{M_2}$$



$$P_M = 2\int_0^{\frac{\pi}{2}} \int_{\frac{2x}{\pi}}^{\sin x} dy\, dx = 2\int_0^{\frac{\pi}{2}} [y]_{\frac{2x}{\pi}}^{\sin x} dx = 2\int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2x}{\pi}\right) dx = 2\left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}} = 2 - \frac{\pi}{2}$$

Problem 1.2 (Theory)

Explain possible physical and geometric interpretation of obtained result.

Math II / Multiple Integrals – 2nd Problem

Problem 2.1 (4 points)

Calculate mass and coordinates of centre of gravity of non-homogeneous lamina in the form of a planar region determined by inequalities $1 \le x^2 + y^2 \le 4$, $y \ge |x|$, if specific density of lamina material is defined by function $\mu(x, y) = x^2 + y^2$. Describe and sketch the region with the calculated centre of gravity.

Solution:

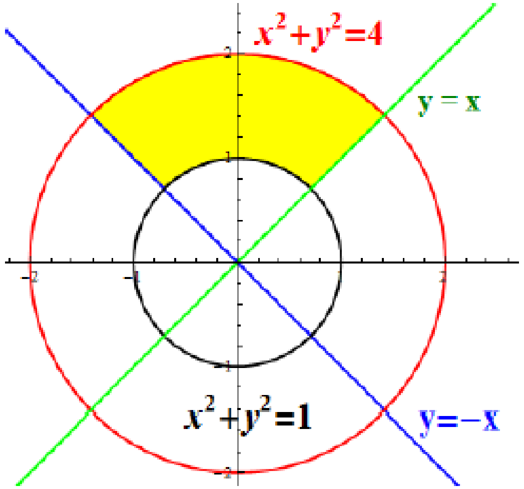
$$M = \left\{ [x, y] \in \boldsymbol{E}^2 : 1 \le x^2 + y^2 \le 4, y \ge |x| \right\}$$

Polar transformation

$$x = \rho \cos \varphi$$

$$y = \rho \sin \varphi \qquad M^* = \left\{ (\rho, \varphi) \in \boldsymbol{E}^2 : 1 \le \rho \le 2, \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \right\}$$

$$J(\rho, \varphi) = \rho$$



$$m_M = \iint_M \mu(x, y)\, dx\, dy = m_{M^*} = \iint_{M^*} (\rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi)\, \rho\, d\rho\, d\varphi = \int_1^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \rho^3 d\rho\, d\varphi = \frac{15\pi}{8}$$

Math II / Multiple Integrals – 2nd Problem

Solution (Continue):

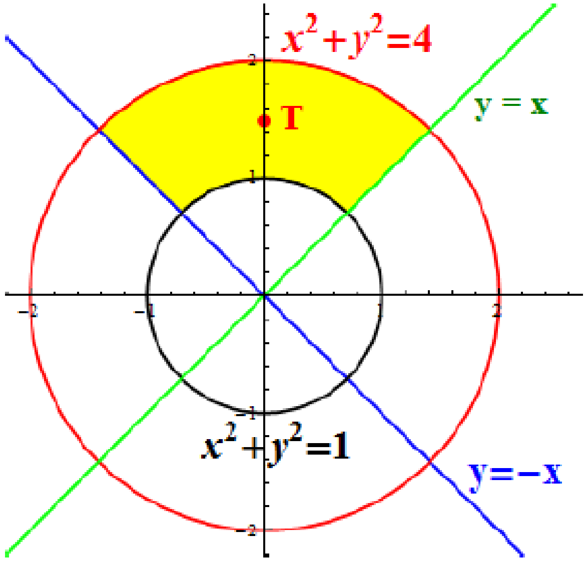
$T=[x_T\,,y_T\,]$

$$x_T=\frac{1}{m_M}\iint_M x\mu(x,y)dxdy=\frac{1}{m_{M^*}}\iint_{M^*}\rho\cos\varphi\,\rho^2\rho d\rho d\varphi=$$

$$=\frac{8}{15\pi}\int_1^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\rho^4\cos\varphi\,d\varphi\,d\rho=0$$

$$y_T=\frac{1}{m_M}\iint_M x\mu(x,y)dxdy=\frac{1}{m_{M^*}}\iint_{M^*}\rho\sin\varphi\,\rho^2\rho d\rho d\varphi=$$

$$=\frac{8}{15\pi}\int_1^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\rho^4\sin\varphi\,d\varphi d\rho=\frac{248\sqrt{2}}{75\pi}$$



$$T=\left[0;\frac{248\sqrt{2}}{75\pi}\right]\doteq[0;1,5]$$

Problem 2.2 (Theory)

Explain properties of double integrals and illustrate on simple examples.

Math II / Multiple Integrals – 3rd Problem

Problem 3.1 (5 points)

Calculate volume of solid bounded by graphs of functions $f(x,y) = 4xy - y^3$ and $g(x,y) = -y$ defined over the region M in the coordinate plane xy, with boundaries in graphs of functions $y = x^3$, $y = \sqrt{x}$. Sketch and describe region M and solid T as set of points.

Solution:

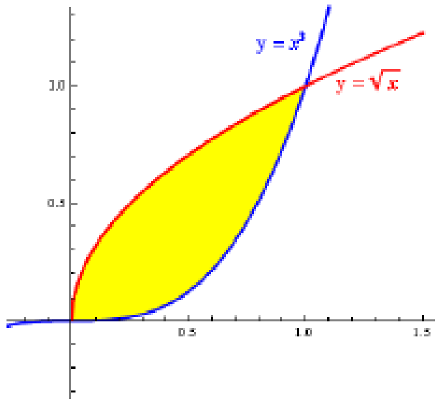
$$M = \{[x,y] \in E^2 : 0 \le x \le 1,\ x^3 \le y \le \sqrt{x}\} \qquad T = \{[x,y,z] \in E^3 : [x,y] \in M,\ -y \le z \le 4xy - y^3\}$$

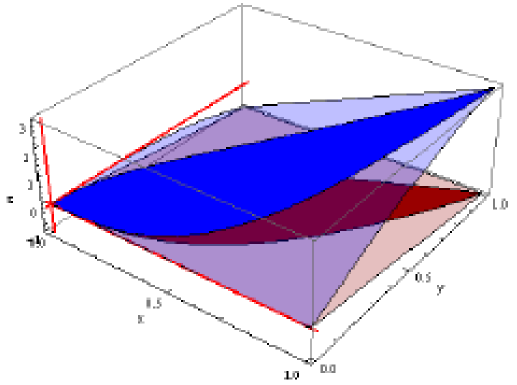
$$x^3 = \sqrt{x} \Leftrightarrow \quad x^6 = x$$

$$x^6 - x = 0$$

$$x(x^5 - 1) = 0$$

$$x = 0 \vee x = 1$$





$$V = \iiint_T 1dxdydz = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-y}^{4xy-y^3} 1dz\, dy\, dx = \frac{145}{273}$$

Problem 3.2 (Theory)

Write and explain Fubini theorem for triple integrals.

Math II / Multiple Integrals – 4th Problem

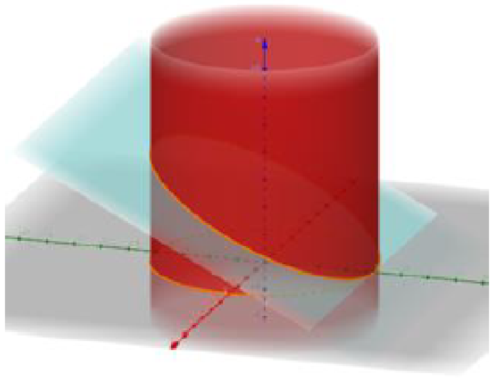
Problem 4.1 (6 points)

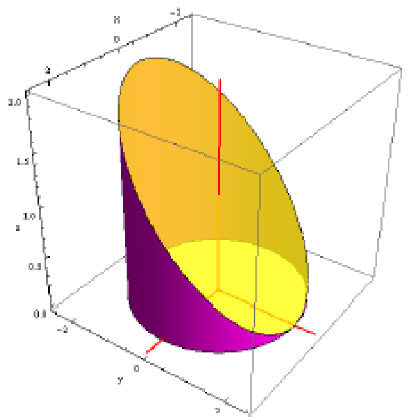
Calculate mass of non-homogeneous solid bounded by cylindrical surface with equation $x^2 + y^2 = 4$ and planes $z = 0$, $y = 2 - 2z$, if specific density of solid material is $\mu(x, y, z) = x + 2$. Sketch and describe solid T.

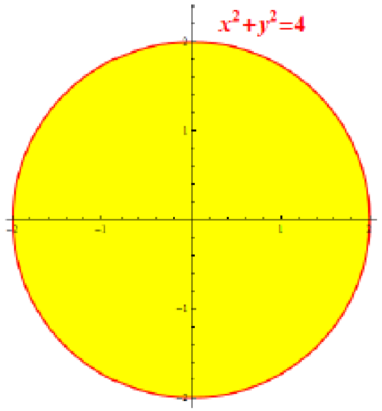
***Solution*:**

$$T = \left\{ [x, y, z] \in E^3 : [x, y] \in R \,,\ 0 \le z \le \frac{2 - y}{2} \right\} \qquad R = \left\{ [x, y] \in E^2 ;\ x^2 + y^2 \le 4 \right\}$$

Cylindrical transformatin







$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$
$$J(\rho, \varphi, z) = \rho$$

$$R^* = \left\{ [\rho, \varphi] \in E^2 ;\ 0 \le \rho \le 2 \,,\ 0 \le \varphi \le 2\pi \right\} \,,\quad T^* = \left\{ [\rho, \varphi, z] \in E^3 : [\rho, \varphi] \in R^* ,\ 0 \le z \le 1 - \frac{\rho \sin \varphi}{2} \right\}$$

$$m = \iiint_T \mu(x, y, z)\, dx\, dy\, dz = \iiint_{T^*} (\rho \cos \varphi + 2)\, \rho\, dz\, d\varphi\, d\rho = \int_0^2 \int_0^{2\pi} \int_0^{1 - \frac{\rho \sin \varphi}{2}} (\rho \cos \varphi + 2)\, \rho\, dz\, d\varphi\, d\rho = 8\pi$$

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Math II / Multiple Integrals – 4th Problem

Problem 4.2 (Theory)

List and explain basic properties of triple integrals.

Math II / Multiple Integrals – 5th Problem

Problem 5.1 (7 points)

Using multiple integration prove validity of the solution of problem posed and solved by Greek mathematician Archimedes (about 287 B. C. - 212 A. D.) who used the exhaustion method (division of solid to small general prisms): **Volume of rotational paraboloid inscribed into rotational cylinder equals one half of the volume of cylinder.**

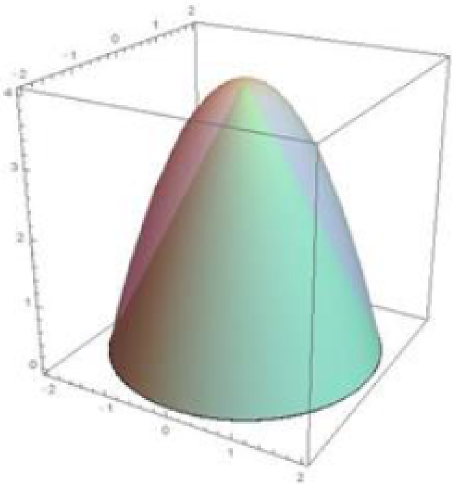
Solution guide:

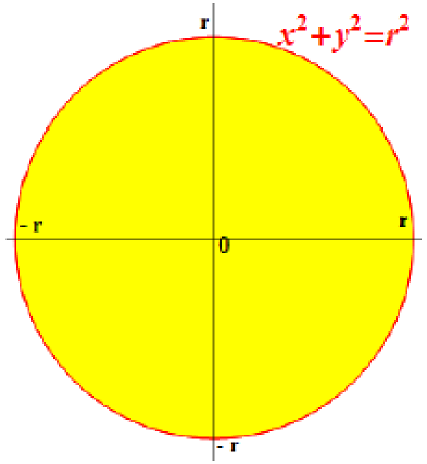
By means of triple integrals express

- Volume of paraboloid V_P (with height h and radius r of its base circle) determined by equation $z = h(1 - \frac{x^2+y^2}{r^2})$
- Volume of cylinder V_V (with height h and radius r of its base circle) determined by equation $z = h$

Math II / Multiple Integrals – 5th Problem

Solution (Continue):





Cylindrical transformation

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$z = z$

$J(\rho,\varphi,z) = \rho$

$$T_P = \left\{[x,y,z]\in E^3 : [x,y]\in R_P,\ 0\le z\le h\left(1-\frac{x^2+y^2}{r^2}\right)\right\} \qquad R_P = \left\{[x,y]\in E^2 ;\ x^2+y^2\le r^2\right\}$$

$$T_V = \left\{[x,y,z]\in E^3 : [x,y]\in R_V,\ 0\le z\le h\right\} \qquad R_V = \left\{[x,y]\in E^2 ;\ x^2+y^2\le r^2\right\}$$

$$R_P^* = \left\{[\rho,\varphi]\in E^2 ;\ 0\le\rho\le r,\ 0\le\varphi\le 2\pi\right\} \ \rightarrow\ T_P^* = \left\{[\rho,\varphi,z]\in E^3 : [\rho,\varphi]\in R_P^*, 0\le z\le h\left(1-\frac{\rho^2}{r^2}\right)\right\}$$

$$R_V^* = \left\{[\rho,\varphi]\in E^2 ;\ 0\le\rho\le r,\ 0\le\varphi\le 2\pi\right\} \ \rightarrow\ T_V^* = \left\{[\rho,\varphi,z]\in E^3 : [\rho,\varphi]\in R_V^*, 0\le z\le h\right\}$$

Math II / Multiple Integrals – 5th Problem

Solution (Continue):

$$V_P = \iiint_{T_P} 1\,dx\,dy\,dz = \iiint_{T_P^*} \rho\,d\rho\,d\varphi\,dz = \iint_{R_P^*} \left(h\left(1-\frac{\rho^2}{r^2}\right) - 0 \right) \rho\,d\varphi\,d\rho = \int_0^r \int_0^{2\pi} h\left(1-\frac{\rho^2}{r^2}\right) \rho\,d\varphi\,d\rho = \frac{\pi h r^2}{2}$$

$$V_V = \iiint_{T_V} 1\,dx\,dy\,dz = \iiint_{T_V^*} (\rho\,d\rho\,d\varphi\,dz = \iint_{R_V^*} (h-0)\,\rho\,d\varphi\,d\rho = \int_0^r \int_0^{2\pi} h\,\rho\,d\varphi\,d\rho = \pi h r^2$$

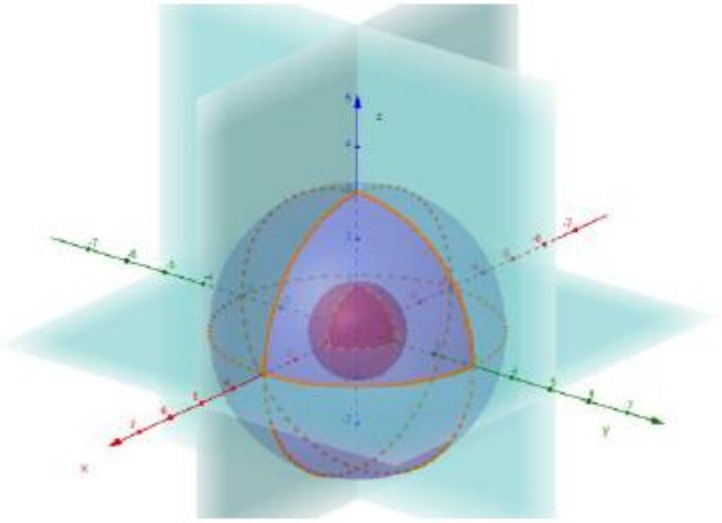
$$\left. \begin{array}{l} V_P = \dfrac{\pi h r^2}{2} \\ \\ V_K = \pi h r^2 \end{array} \right\} \frac{V_P}{V_V} = \frac{\frac{\pi h r^2}{2}}{\pi h r^2} = \frac{1}{2} \Rightarrow V_P = \frac{1}{2} V_V$$

Math II / Multiple Integrals – 5th Problem

Problem 5.2 (Theory)

Write equations of integral transformations of triple integrals to spherical coordinates.

Illustrate on example of calculation of volume of a solid formed by parts of concentric spheres with centres in origin of coordinate system and radii 1 and 3 and half-planes $x \geq 0, y \geq 0, z \geq 0$



Math School

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Pythagoras
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